Lesson 10
Transforming 3D Integrals
Example 1: Triple Integrals to Compute Volume

Recall that in previous chapters we could find the length of an interval $I$ by computing $\int dx$ or the area of a region $R$ by computing $\int \int dA$.

It follows that we can compute the volume of a 3-dimensional region $R$ by calculating $\int \int \int 1 \, dV$.

For this cube, the calculation is not exciting:

$$\int \int \int 1 \, dx \, dy \, dz = \int \int \left[ x \right]_{x=-1}^{x=1} \, dy \, dz$$

$$= \int \left[ 2y \right]_{y=0}^{y=2} \, dz$$

$$= \left[ 4z \right]_{z=1}^{z=3}$$

$$= 8$$

$-1 \leq x \leq 1, \ 0 \leq y \leq 2, \ 1 \leq z \leq 3$
Example 2: Triple Integrals to Compute Volume

We can spice things up a bit. We can find the volume of the region above the xy-plane, the xz-plane, the yz-plane, and below the plane given by \( x + y + 4z = 8 \):

If \( x + y + 4z = 8 \), then \( z = \frac{8 - x - y}{4} \). So z-top is the plane \( z = \frac{8 - x - y}{4} \) and z-bottom is the plane \( z = 0 \).

Find the intersection between \( z = \frac{8 - x - y}{4} \) and \( z = 0 \): We get the line \( y = 8 - x \), which is our y-top. Our y-bottom is the line \( y = 0 \).

Finally, integrate \( x \) from \( x = 0 \) to \( x = 8 \).
Example 2: Triple Integrals to Compute Volume

We can spice things up a bit. We can find the volume of the region above the xy-plane, the xz-plane, the yz-plane, and below the plane given by $x + y + 4z = 8$:

$$
\iiint_{0}^{8} \frac{8-x-y}{4} \, dz \, dy \, dx
= \frac{1}{4} \iiint_{0}^{8} (8-x-y) \, dy \, dx
= \frac{1}{4} \left[ \int_{0}^{8} \left( 8y - xy - \frac{y^2}{2} \right) \, dy \right]_{y=0}^{8-x}
= \frac{1}{4} \left( 8(8-x) - x(8-x) - \frac{(8-x)^2}{2} \right) \, dx
= \frac{64}{3}
$$

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Example 3: Changing the Order of Integration

Repeat Example 2 by computing \( \iiint_R dy \, dx \, dz \) where \( R \) is the region above the xy-plane, the xz-plane, the yz-plane, and below the plane given by \( x + y + 4z = 8 \)

If \( x + y + 4z = 8 \), then \( y = 8 - x - 4z \). So y-top is \( 8 - x - 4z \) and y-bottom is the plane \( y = 0 \).

\( x + y + 4z = 8 \) and \( y = 0 \) intersect at the line \( x = 8 - 4z \) (our x-top). Our x-bottom is the line \( x = 0 \).

Finally, integrate z from \( z = 0 \) to \( z = 2 \).

\[
\int_0^2 \int_0^{8-4z} \int_0^{8-x-4z} 1 \, dy \, dx \, dz = \frac{64}{3}
\]
Example 4: 3D-Change of Variables

Compute the volume of the parallelepiped described by the region \( R_{xyz} \) between the planes \( z = 3x \) and \( z = 3x + 2 \), \( y = x \) and \( y = x + 4 \), and \( y = -2x \) and \( y = -2x + 3 \).

First, pick a clever change of variables:

\[
\begin{align*}
    u &= z - 3x, \text{ and let } u \text{ run from 0 to 2} \\
    v &= y - x, \text{ and let } v \text{ run from 0 to 4} \\
    w &= y + 2x, \text{ and let } w \text{ run from 0 to 3}
\end{align*}
\]

Our integral is much more manageable now:

\[
\iiint_{R_{xyz}} 1 \, dx \, dy \, dz = \iiint_{R_{uvw}} \left| V_{xyz}(u, v, w) \right| \, du \, dv \, dw
\]

\[
V_{xyz}(u, v, w) = \left| \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \right|
\]

\[
\iiint_{0,0,0}^{3,4,2} \left| V_{xyz}(u, v, w) \right| \, du \, dv \, dw
\]
The Volume Conversion Factor:

\[ \iiint 1 \, dx \, dy \, dz = \iiint |V_{xyz}(u, v, w)| \, du \, dv \, dw \]

Let \( T(u, v, w) \) be a transformation from uvw-space to xyz-space. That is, \( T(u,v,w) = (T_1(u,v,w), T_2(u,v,w), T_3(u,v,w)) = (x(u,v,w), y(u,v,w), z(u,v,w)) \).

Then \( V_{xyz}(u, v, w) = \left| \begin{array}{ccc} \frac{\partial T_1}{\partial u} & \frac{\partial T_2}{\partial u} & \frac{\partial T_3}{\partial u} \\ \frac{\partial T_1}{\partial v} & \frac{\partial T_2}{\partial v} & \frac{\partial T_3}{\partial v} \\ \frac{\partial T_1}{\partial w} & \frac{\partial T_2}{\partial w} & \frac{\partial T_3}{\partial w} \end{array} \right| \]

Or \( V_{xyz}(u, v, w) = \left| \begin{array}{ccc} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{array} \right| \)

Note: Just like \( A_{xy}(u, v) \), we need \( V_{xyz}(u, v, w) \) to be positive. Hence, the absolute value bars in the formula above.
Example 4: 3D-Change of Variables

Compute the volume of the parallelepiped described by the region $R_{xyz}$ between the planes $z = 3x$ and $z = 3x + 2$, $y = x$ and $y = x + 4$, and $y = -2x$ and $y = -2x + 3$.

$$u = z - 3x$$
$$v = y - x$$
$$w = y + 2x$$

$$x = \frac{w - v}{3}$$
$$y = \frac{2v + w}{3}$$
$$z = u + w - v$$

$$\int \int \int_{V_{xyz}} (u, v, w) \, du \, dv \, dw = \int \int \int_{V_{xyz}} \frac{1}{3} \, du \, dv \, dw$$

$$= 8$$

Use $\frac{1}{3}$. 

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Example 4: 3D-Change of Variables

Compute the volume of the parallelepiped described by the region $R_{xyz}$ between the planes $z = 3x$ and $z = 3x + 2$, $y = x$ and $y = x + 4$, and $y = -2x$ and $y = -2x + 3$.

No need to study this, it's just nice to verify this works...

Check:
For a parallelepiped generated by three intersecting vectors $X$, $Y$ and $Z$, $V_{\text{parallelepiped}} = (X \times Y) \cdot Z$

Parallelepiped is generated by vectors $(1,1,3)$, $\left(-\frac{4}{3}, \frac{8}{3}, -4\right)$, and $(0,0,2)$.

$V_{\text{parallelepiped}} = \left((1,1,3) \times \left(-\frac{4}{3}, \frac{8}{3}, -4\right)\right) \cdot (0,0,2) = 8$
Example 5: A Mathematica-Assisted Change of Variables

Find the volume of the "football" whose outer skin is described by the parametric equation \( F(s, t) = (0, 0, 4s) + \left(4 - s^2\right)\left(\cos(t), \sin(t), 0\right) \) for \(-2 \leq s \leq 2\) and \(0 \leq t \leq 2\pi\).

First, fill in the football:

\[
F(r, s, t) = (0, 0, 4s) + r\left(4 - s^2\right)\left(\cos(t), \sin(t), 0\right)
\]

\(-2 \leq s \leq 2\)

\(0 \leq t \leq 2\pi\)

\(0 \leq r \leq 1\)

Change of variables:

\[
x(r, s, t) = r\left(4 - s^2\right)\cos(t)
\]

\[
y(r, s, t) = r\left(4 - s^2\right)\sin(t)
\]

\[
z(r, s, t) = 4s
\]

\[
\int_{0}^{2\pi} \int_{0}^{2} \int_{0}^{1} |V_{xyz}(r, s, t)| \, dr \, ds \, dt
\]
Example 5: A Mathematica-Assisted Change of Variables

Find the volume of the "football" whose outer skin is described by the parametric equation $F(s, t) = (0, 0, 4s) + (4 - s^2)(\cos(t), \sin(t), 0)$ for $-2 \leq s \leq 2$ and $0 \leq t \leq 2\pi$.

$$V_{xyz}(r, s, t) = \left| \begin{array}{ccc} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{array} \right|$$

$$= -4(16r - 8rs^2 + rs^4) \quad \text{(Mathematica)}$$

$$\int_{0}^{2\pi} \int_{-2}^{2} \int_{0}^{1} V_{xyz}(r, s, t) \, dr \, ds \, dt = \frac{2048}{15} \pi \quad \text{(Mathematica)}$$

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Example 5: A Mathematica-Assisted Change of Variables

Find the volume of the "football" whose outer skin is described by the parametric equation \( F(s, t) = (0, 0, 4s) + (4 - s^2)(\cos(t), \sin(t), 0) \) for \(-2 \leq s \leq 2\) and \(0 \leq t \leq 2\pi\).

Check using solids of revolution:

\[
\pi \int_{-8}^{8} \left( 4 - \left( \frac{x}{4} \right)^2 \right)^2 \, dx = \frac{2048}{15} \pi
\]
Example 6: Beyond Volume Calculations

It is easy to see that \( \iiint_{R} dV \) computes the volume of a solid. But it's harder to interpret \( \iiint_{R} f(x, y, z) \, dV \) since \( f(x, y, z) \) lives in 4 dimensions.

We need an example to keep referring back to to give us some intuition.

A cube of varying density has its density at each point \((x, y, z)\) described by \( f(x, y, z) = x^2 y^4 \, \text{g/cm}^3 \). Find the mass of the cube.

\[
\iiint_{V} x^2 y^4 \, dx \, dy \, dz = \frac{128}{15} \text{ grams}
\]
3D Integrals:

\[ \iiint_{R_{xyz}} f(x, y, z) \, dx \, dy \, dz = \iiint_{R_{uvw}} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| V_{xyz}(u, v, w) \right| \, du \, dv \, dw \]

\[ V_{xyz}(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} \]

In my opinion, a good way to think about \( \iiint_{R_{xyz}} f(x, y, z) \, dx \, dy \, dz \) is as a calculation of the mass of \( R_{xyz} \) where \( f(x, y, z) \) is the density of the solid at any given point \((x, y, z)\).
Example 7: A 3D-Change of Variables with an Integrand

Compute $\iiint_{R_{xyz}} 9y \, dx \, dy \, dz$ where $R_{xyz}$ is the parallelepiped that is between the planes $z = 3x$ and $z = 3x + 2$, $y = x$ and $y = x + 4$, and $y = -2x$ and $y = -2x + 3$.

All from Example 4:

- $u = z - 3x$, $0 \leq u \leq 2$
- $v = y - x$, $0 \leq v \leq 4$
- $w = y + 2x$, $0 \leq w \leq 3$
- $x = \frac{w - v}{3}$
- $y = \frac{2v + w}{3}$
- $z = u + w - v$
- $|V_{xyz}(u, v, w)| = \frac{1}{3}$

\[
\iiint_{R_{xyz}} 9y \, dx \, dy \, dz = \int_{0}^{3} \int_{0}^{4} \int_{0}^{2} 9y(u, v, w) \left| V_{xyz}(u, v, w) \right| \, du \, dv \, dw \\
= \int_{0}^{3} \int_{0}^{4} \int_{0}^{2} 9 \left( \frac{2v + w}{3} \right) \frac{1}{3} \, du \, dv \, dw \\
= \int_{0}^{3} \int_{0}^{4} \int_{0}^{2} \left( 2v + w \right) \, du \, dv \, dw = 132
\]
Example 8: A Traditional Triple Integral

Set up, but do not compute \( \iiint_{R_{xyz}} 2y \, dx \, dy \, dz \) where \( R \) is the paraboloid \( z = 9 - x^2 - y^2 \) such that \( z > 0 \).

\[
\iiint_{R_{xyz}} 2y \, dz \, dy \, dx = \int_{\sqrt[3]{9-x^2}}^{3} \int_{\sqrt[3]{9-x^2}}^{9-x^2} \int_{0}^{2y} 2y \, dz \, dy \, dx
\]

- **z-top** is the paraboloid \( z = 9 - x^2 - y^2 \) and **z-bottom** is the plane \( z = 0 \).

- **Find the intersection between** \( z = 9 - x^2 - y^2 \) and \( z = 0 \): We get the circle \( x^2 + y^2 = 9 \), so our **y-top** is \( y = \sqrt{9 - x^2} \) and our **y-bottom** is \( y = -\sqrt{9 - x^2} \).

- **Finally, integrate** \( x \) from \( x = -3 \) to \( x = 3 \).