Lesson 5

Double Integrals, Volume Calculations, and the Gauss-Green Formula
Example 1: A Double Integral Over a Rectangular Region

Let \( f(x, y) = \sin(y) + \cos(x) + 2 \) and let \( R \) be the region in the xy-plane bounded by \( x = 0, x = \pi, y = -2\pi, \) and \( y = \pi. \) Calculate \( \iint_R f(x, y) \, dA : \)

\[
\int_{-2\pi}^{\pi} \int_{0}^{\pi} \left( \sin(y) + \cos(x) + 2 \right) \, dy \, dx
\]

\[
= \int_{0}^{\pi} \left[ -\cos(y) + y \cos(x) + 2y \right]_{y=-2\pi}^{y=\pi} \, dx
\]

\[
= \int_{0}^{\pi} \left[ 2 + 6\pi + 3\pi \cos(x) \right] \, dx
\]

\[
= \left[ 2x + 6\pi x + 3\pi \sin(x) \right]_{x=0}^{x=\pi}
\]

\[
= 2\pi + 6\pi^2
\]
Example 2: Changing the Order of Integration

Calculate $\iint_R f(x, y) \, dA$ by computing $\iint_R f(x, y) \, dx \, dy$ instead of $\iint_R f(x, y) \, dy \, dx$.

$$
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (\sin(y) + \cos(x) + 2) \, dx \, dy
$$

$$
= \int_{-\pi}^{\pi} \left[ x \sin(y) + \sin(x) + 2x \right]_{x=0}^{x=\pi} \, dy
$$

$$
= \int_{-\pi}^{\pi} 2\pi + \pi \sin(y) \, dy
$$

$$
= \left[ 2\pi y - \pi \cos(y) \right]_{y=-\pi}^{y=\pi}
$$

$$
= 2\pi + 6\pi^2
$$
Computing Double Integrals Over a Rectangular Region

Let $z = f(x, y)$ be a surface and let $R$ be the rectangular region in the $xy$-plane bounded by $a \leq x \leq b$ and $c \leq y \leq d$. Calculate $\int \int_R f(x, y) \, dA$:

1) Set up $\int \int_R f(x, y) \, dx \, dy$.

2) Hold $y$ constant and integrate with respect to $x$:
$$\int_c^d \left[ F(x, y) \right]_{x=a}^{x=b} \, dy = \int_c^d \left[ F(b, y) - F(a, y) \right] \, dy$$

3) $F(b, y) - F(a, y)$ is a function of $y$ that measures the area of horizontal cross sections between $y = c$ and $y = d$. So we can integrate these area cross sections to get our volume:
$$\int_c^d \left[ F(b, y) - F(a, y) \right] \, dy$$

1) Set up $\int \int_R f(x, y) \, dy \, dx$.

2) Hold $x$ constant and integrate with respect to $y$:
$$\int_a^b \left[ G(x, y) \right]_{y=c}^{y=d} \, dx = \int_a^b \left[ G(x, d) - G(x, c) \right] \, dx$$

3) $G(x, d) - G(x, c)$ is a function of $x$ that measures the area of vertical cross sections between $x = a$ and $x = b$. So we can integrate these area cross sections to get our volume:
$$\int_a^b \left[ G(x, d) - G(x, c) \right] \, dx$$
Let \( f(x, y) = 2\sin(xy) - 1 \) and let \( R \) be the region in the \( xy \)-plane bounded by \( x = -4, x = 4, y = -3, \) and \( y = 3 \). Calculate \( \iint_R f(x, y) \, dA : \)

\[
\begin{align*}
\int_{-3}^{3} \int_{-4}^{4} (2\sin(xy) - 1) \, dx \, dy &= \int_{-3}^{3} \left[ \frac{-2\cos(xy)}{y} - x \right]_{x=-4}^{x=4} \, dy \\
&= \int_{-3}^{3} \left[ \left( -\frac{2\cos(4y)}{y} - 4 \right) - \left( -\frac{2\cos(-4y)}{y} - (-4) \right) \right] \, dy \\
&= \int_{-3}^{3} -8 \, dy \\
&= \left[ -8y \right]_{y=-3}^{y=3} \\
\end{align*}
\]

How did this end up negative??
Example 4: A Double Integral Over a Non-Rectangular Region

Let \( f(x, y) = x^2 + y^2 + 3 \) and let \( R \) be the region in the xy-plane bounded above by \( y = -x^2 + 4 \) and below by \( y = -5 \). Calculate \( \iint_R f(x, y) \, dA \):

\[
\iint_R (x^2 + y^2 + 3) \, dy \, dx
\]

where \( x_{\text{left}} = -3 \), \( x_{\text{right}} = 3 \), \( y_{\text{high}}(x) = -x^2 + 4 \), and \( y_{\text{low}}(x) = -5 \).

\[
= \int_{-3}^{3} \left[ \int_{-5}^{-x^2+4} (x^2 + y^2 + 3) \, dy \right] \, dx
\]

\[
= \int_{-3}^{3} \left[ xy^2 + \frac{y^3}{3} + 3y \right]_{y=-5}^{y=-x^2+4} \, dx
\]

\[
= \int_{-3}^{3} \left( 90 - 10x^2 + 3x^4 - \frac{x^6}{3} \right) \, dx
\]

\[
y_{\text{high}}(x) = -x^2 + 4
\]

\[
y_{\text{low}}(x) = -5
\]

\[
x_{\text{left}} = -3
\]

\[
x_{\text{right}} = 3
\]

\[
90 - 10x^2 + 3x^4 - \frac{x^6}{3} \text{ is a function that sweeps out vertical area cross sections from } -3 \text{ to } 3.
\]
Example 4: A Double Integral Over a Non-Rectangular Region

Let \( f(x, y) = x^2 + y^2 + 3 \) and let \( R \) be the region in the xy-plane bounded above by \( y = -x^2 + 4 \) and below by \( y = -5 \). Calculate \( \iint_R f(x, y) \, dA \):

If we integrate these cross sections given by \( 90 - 10x^2 + 3x^4 - \frac{x^6}{3} \) from \( x = -3 \) to \( x = 3 \), we accumulate the volume of the solid:

\[
\begin{align*}
\int_{-3}^{3} 
\left( 90 - 10x^2 + 3x^4 - \frac{x^6}{3} \right) dx
\end{align*}
\]

\[
\begin{align*}
= & \left[ 90x - \frac{10}{3}x^3 + \frac{3}{5}x^5 - \frac{x^7}{21} \right]_{x=-3}^{x=3} \\
= & \frac{15516}{35}
\end{align*}
\]
Calculate \( \iint_{R} f(x, y) \, dA \) again, but change the order of integration:

\[
\int_{y_{\text{bottom}}}^{y_{\text{top}}} \int_{x_{\text{low}}(y)}^{x_{\text{high}}(y)} \left( x^2 + y^2 + 3 \right) \, dx \, dy
\]

\[
= \int_{-5}^{4} \int_{-\sqrt{4-y}}^{\sqrt{4-y}} \left( x^2 + y^2 + 3 \right) \, dx \, dy
\]

\[
= \int_{-5}^{4} \left[ \frac{x^3}{3} + y^2x + 3x \right]_{x=-\sqrt{4-y}}^{x=\sqrt{4-y}} \, dy
\]

\[
= \int_{-5}^{4} \left( \frac{26\sqrt{4-y}}{3} \right. - \frac{2}{3} y\sqrt{4-y} + 2y^2 \sqrt{4-y} \) \, dy
\]

\[
y_{\text{top}} = 4 \\
x_{\text{low}}(y) = -\sqrt{4-y} \\
x_{\text{high}}(y) = \sqrt{4-y}
\]

26\sqrt{4-y} - \frac{2}{3} y\sqrt{4-y} + 2y^2 \sqrt{4-y} is a function that sweeps out horizontal area cross sections from \( y = -5 \) to \( y = 4 \).
Example 5: Change the Order of Integration

Let \( f(x, y) = x^2 + y^2 + 3 \) and let \( R \) be the region in the xy-plane bounded above by \( y = -x^2 + 4 \) and below by \( y = -5 \). Calculate \( \iint f(x, y) \, dA \):

\[
\frac{26}{3} \sqrt{4 - y} - \frac{2}{3} y \sqrt{4 - y} + 2y^2 \sqrt{4 - y}
\]

is a function that sweeps out horizontal area cross sections from \( y = -5 \) to \( y = 4 \).

\[
\int_{-5}^{4} \left( \frac{26}{3} \sqrt{4 - y} - \frac{2}{3} y \sqrt{4 - y} + 2y^2 \sqrt{4 - y} \right) \, dy
\]

\[
= \left[ -\frac{4}{105} (4 - y)^{3/2} \left( 15y^2 + 41y + 261 \right) \right]_{y=-5}^{y=4}
\]

\[
= 15516
\]

\[
= \frac{15516}{35}
\]
Summary of Example 4 & 5

\[ \int_{-3}^{3} \int_{-5}^{4-y} \left( x^2 + y^2 + 3 \right) \, dy \, dx \quad \text{and} \quad \int_{-5}^{\sqrt{4-y}} \int_{-3}^{4} \left( x^2 + y^2 + 3 \right) \, dx \, dy \]

compute the exact same thing!!

We have changed the order of integration. This works whenever we can bound the curve with top/bottom bounding functions or left/right bounding functions.

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Computing Double Integrals Over a Non-Rectangular Region

Let \( z = f(x, y) \) be a surface and let \( R \) be the non-rectangular region in the xy-plane bounded by \( y = f(x) \) on top and \( y = g(x) \) on bottom. Calculate \( \iint f(x, y) \, dA \):

1) Set up \( \int_{x_{\text{left}}}^{x_{\text{right}}} \int_{y_{\text{low}}(x)}^{y_{\text{high}}(x)} f(x, y) \, dy \, dx \)

2) Hold \( x \) constant and integrate with respect to \( y \):
\[
\int_{x_{\text{left}}}^{x_{\text{right}}} \left[ \int_{y_{\text{low}}(x)}^{y_{\text{high}}(x)} f(x, y) \, dy \right] \, dx = \int_{x_{\text{left}}}^{x_{\text{right}}} \left[ F(x, y_{\text{high}}(x)) - F(x, y_{\text{low}}(x)) \right] \, dx
\]

3) \( \left[ F(x, y_{\text{high}}(x)) - F(x, y_{\text{low}}(x)) \right] \) is a function of \( x \) that measures the area of vertical cross sections between \( x = x_{\text{left}} \) and \( x = x_{\text{right}} \).

So we can integrate these area cross sections to get our volume:
\[
\int_{x_{\text{left}}}^{x_{\text{right}}} \left[ F(x, y_{\text{high}}(x)) - F(x, y_{\text{low}}(x)) \right] \, dx
\]

Likewise for the order integration being reversed.
Example 6: Using the Double Integral to Find the Area of a Region

Let $R$ be the region in the xy-plane from Example 4 and 5 that is bounded above by $y = -x^2 + 4$ and below by $y = -5$. Find the area of $R$.

**Claim:** For a region $R$, \( \text{Area}_R = \iint 1 \, dA \)

**Old Way:**

\[
\int_{-3}^{3} \left( (-x^2 + 4) - (-5) \right) \, dx = \int_{-3}^{3} (-x^2 + 9) \, dx = \left[ -\frac{x^3}{3} + 9x \right]_{x=-3}^{x=3} = 36
\]

**New Way:**

\[
\iint 1 \, dA = \int_{-3}^{3} \int_{y=-x^2+4}^{y=-5} 1 \, dy \, dx = \int_{-3}^{3} \left[ y \right]_{y=-x^2+4}^{y=-5} \, dx = \int_{-3}^{3} \left( (-x^2 + 4) - (-5) \right) \, dx = 36
\]

It works!!!
The Gauss-Green Formula

Let $R$ be a region in the $xy$-plane whose boundary is parameterized by $(x(t), y(t))$ for $t_{\text{low}} \leq t \leq t_{\text{high}}$. Then the following formula holds:

$$
\iint_{R} \left( \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} \right) dA = \int_{t_{\text{low}}}^{t_{\text{high}}} \left( m(x(t), y(t)) x'(t) + n(x(t), y(t)) y'(t) \right) dt
$$

Main Idea: You can compute a double integral as a single integral, as long as you have a reasonable parameterization of the boundary curve of the region $R$.

Main Rule: For this formula to hold:

1) Your parameterization must be counterclockwise.
2) Your $t_{\text{low}} \leq t \leq t_{\text{high}}$ must bring you around the boundary curve exactly ONCE.
3) Your boundary curve must be a simple closed curve.
The Gauss-Green formula is also known as the Gauss-Green Theorem or simply Green’s Theorem.

Let $R$ be a region in the $xy$-plane whose boundary is parameterized by $(x(t),y(t))$ for $t_{\text{low}} \leq t \leq t_{\text{high}}$. Then the following formula holds:

$$
\iint_R \left( \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} \right) \, dA = \int_{t_{\text{low}}}^{t_{\text{high}}} \left( m(x(t), y(t)) x'(t) + n(x(t), y(t)) y'(t) \right) \, dt
$$

Main Idea: You can compute a double integral as a single integral, as long as you have a reasonable parameterization of the boundary curve of the region $R$.

Main Rule: For this formula to hold:

1) Your parameterization must be counterclockwise.
2) Your $t_{\text{low}} \leq t \leq t_{\text{high}}$ must bring you around the boundary curve exactly ONCE.
3) Your boundary curve must be a simple closed curve.
Example 6: Let \( f(x, y) = y - xy + 9 \) and let \( R \) by the region in the \( xy \)-plane whose boundary is described by the ellipse \( \left( \frac{x}{3} \right)^2 + \left( \frac{y}{4} \right)^2 = 1 \).

Find \( \iint_R f(x, y) \, dx \, dy \):

\[
\text{Gauss-Green: } \iint_R \left( \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} \right) \, dx \, dy = \int_{t_{\text{low}}}^{t_{\text{high}}} \left( m(x(t), y(t)) \, x'(t) + n(x(t), y(t)) \, y'(t) \right) \, dt
\]

Let \( (x(t), y(t)) = (3 \cos(t), 4 \sin(t)) \) for \( 0 \leq t \leq 2\pi \).

Let \( f(x, y) = \frac{\partial n}{\partial x} \) and \( 0 = \frac{\partial m}{\partial y} \).

Then \( n(x, y) = \int_0^x f(s, y) \, ds = \int_0^x (y - sy + 9) \, ds = \left[ sy - \frac{s^2y}{2} + 9s \right]_0^x \]

\[= xy - \frac{x^2y}{2} + 9x \]

So \( n(x(t), y(t)) = 12\sin(t) \cos(t) - 18\cos^2(t) \sin(t) + 27 \cos(t) \)

Further, \( (x'(t), y'(t)) = (-3\sin(t), 4 \cos(t)) \).

Plug in: \( \int_{t_{\text{low}}}^{t_{\text{high}}} \left( m(x(t), y(t)) \, x'(t) + n(x(t), y(t)) \, y'(t) \right) \, dt \)
Example 6: Let \( f(x, y) = y - xy + 9 \) and let \( R \) be the region in the \( xy \)-plane whose boundary is described by the ellipse \( \left( \frac{x}{3} \right)^2 + \left( \frac{y}{4} \right)^2 = 1 \).

Plug into Gauss-Green:

\[
\int_{t_{low}}^{t_{high}} \left( m(x(t), y(t)) x'(t) + n(x(t), y(t)) y'(t) \right) dt
\]

\[
= \int_0^{2\pi} \left( 0 \cdot (-3\sin(t)) + \left( 12\sin(t)\cos(t) - 18\cos^2(t)\sin(t) + 27\cos(t) \right) (4\cos(t)) \right) dt
\]

\[
= \int_0^{2\pi} \left( 48\cos^2(t)\sin(t) - 72\cos^3(t)\sin(t) + 108\cos^2(t) \right) dt
\]

\[
= 108\pi
\]

Hint: \( \int_a^b \sin^k(t)\cos(t)dt = \frac{\sin^{k+1}(b)}{k+1} - \frac{\sin^{k+1}(a)}{k+1} \)

and \( \int_a^b \cos^k(t)\sin(t)dt = \frac{-\cos^{k+1}(b)}{k+1} + \frac{\cos^{k+1}(a)}{k+1} \)

and \( \cos^2(t) = \frac{1 + \cos(2t)}{2} \)
Process for Using Gauss-Green to Compute Integrals

Let $z = f(x, y)$ be a surface and let $R$ be a region in the $xy$-plane parameterized by $(x(t), y(t))$ for $t_{\text{low}} \leq t \leq t_{\text{high}}$. Calculate $\iint_R f(x, y) \, dA$:

1) Gauss-Green: $\iint_R \left( \frac{\partial n}{\partial x} \frac{\partial m}{\partial y} - \frac{\partial m}{\partial x} \frac{\partial n}{\partial y} \right) \, dx \, dy = \int_{t_{\text{low}}}^{t_{\text{high}}} \left( m(x(t), y(t)) x'(t) + n(x(t), y(t)) y'(t) \right) \, dt$

2) Substitute as follows:

| $m(x(t), y(t))$ | 0 |
| $n(x(t), y(t))$ | $\int_0^x f(s, y) \, ds$ |
| $[t_{\text{low}}, t_{\text{high}}]$ | Bounds for $t$ in $(x(t), y(t))$ |
| $x'(t)$ | Derivative of $x(t)$ |
| $y'(t)$ | Derivative of $y(t)$ |

3) Crunch the integral! Don't forget your trig identities.
Proof of Gauss-Green Theorem

Let's just prove this for one piece: Show

\[
\iint_R \left( -\frac{\partial m}{\partial y} \right) dA = \int_{t_{\text{low}}}^{t_{\text{high}}} \left( m(x(t), y(t)) x'(t) + n(x(t), y(t)) y'(t) \right) dt
\]

Let \( y_{\text{high}}(x) \) be a function of \( x \) that traces the boundary of the top half of \( R \) and let \( y_{\text{low}}(x) \) trace the boundary of the bottom half of \( R \).

Let \((x(t), y(t))\) be a counterclockwise parameterization of the boundary of \( R \) for \( t_{\text{low}} < t < t_{\text{high}} \) that traverses the boundary only once:

Then \((x(t), y(t))\) for \( t_{\text{low}} \leq t \leq t_{\text{mid}} \) traces out \( y_{\text{high}}(x) \) from right to left.

Also \((x(t), y(t))\) for \( t_{\text{mid}} \leq t \leq t_{\text{high}} \) traces out \( y_{\text{low}}(x) \) from left to right.
Proof of Gauss-Green Theorem

Prove: \[ \iint_R \left( \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} \right) \, dA = \int_{t_{\text{low}}}^{t_{\text{high}}} \left( m(x(t), y(t)) x'(t) + n(x(t), y(t)) y'(t) \right) \, dt \]

\[ \iint_R \frac{\partial m}{\partial y} \, dA = \int_a^b y_{\text{high}}(x) \frac{\partial m}{\partial y} \, dy \, dx \]

\[ = \int_a^b \left[ m(x, y_{\text{high}}(x)) - m(x, y_{\text{low}}(x)) \right] \, dx \]

\[ = \int_a^b m(x, y_{\text{high}}(x)) \, dx - \int_a^b m(x, y_{\text{low}}(x)) \, dx \]

\[ = \int_{t_{\text{mid}}}^{t_{\text{high}}} m(x(t), y(t)) \, \frac{dx}{dt} \, dt - \int_{t_{\text{mid}}}^{t_{\text{high}}} m(x(t), y(t)) \, \frac{dx}{dt} \, dt \]
Proof of Gauss-Green Theorem

Prove: \[ \int \int_R \left( \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} \right) \, dA = \int_{t_{\text{low}}}^{t_{\text{high}}} \left( m(x(t), y(t)) x'(t) + n(x(t), y(t)) y'(t) \right) \, dt \]

\[ = \int_{t_{\text{low}}}^{t_{\text{mid}}} m(x(t), y(t)) \frac{dx}{dt} \, dt - \int_{t_{\text{mid}}}^{t_{\text{high}}} m(x(t), y(t)) \frac{dx}{dt} \, dt \]

\[ = \int_{t_{\text{low}}}^{t_{\text{mid}}} m(x(t), y(t)) x'(t) \, dt - \int_{t_{\text{mid}}}^{t_{\text{high}}} m(x(t), y(t)) x'(t) \, dt \]

\[ = - \int_{t_{\text{low}}}^{t_{\text{mid}}} m(x(t), y(t)) x'(t) \, dt - \int_{t_{\text{mid}}}^{t_{\text{high}}} m(x(t), y(t)) x'(t) \, dt \]

\[ = - \int_{t_{\text{low}}}^{t_{\text{high}}} m(x(t), y(t)) x'(t) \, dt \]

Hence \[ \int \int_R \left( - \frac{\partial m}{\partial y} \right) \, dA = \int_{t_{\text{low}}}^{t_{\text{high}}} m(x(t), y(t)) x'(t) \, dt \]
Proof of Gauss-Green Theorem

Prove: \[ \iiint_R \left( \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} \right) dA = \int_{t_{low}}^{t_{high}} \left( m(x(t), y(t)) x'(t) + n(x(t), y(t)) y'(t) \right) dt \]

Hence \[ \iint_R \left( - \frac{\partial m}{\partial y} \right) dA = \int_{t_{low}}^{t_{high}} m(x(t), y(t)) x'(t) dt \]

We can likewise prove... \[ \iint_R \left( \frac{\partial n}{\partial x} \right) dA = \int_{t_{low}}^{t_{high}} \left( n(x(t), y(t)) y'(t) \right) dt \]

Put it Together:

\[ \iiint_R \left( \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} \right) dA = \iint_R \frac{\partial n}{\partial x} dA - \iint_R \frac{\partial m}{\partial y} dA \]

\[ = \int_{t_{low}}^{t_{high}} n(x(t), y(t)) y'(t) dt + \int_{t_{low}}^{t_{high}} m(x(t), y(t)) x'(t) dt \]

\[ = \int_{t_{low}}^{t_{high}} \left( m(x(t), y(t)) x'(t) + n(x(t), y(t)) y'(t) \right) dt \]

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When Should I Use What??

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<th>Region Bounded</th>
<th>Integrate with respect to</th>
<th>Integrate with respect to</th>
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<tr>
<td>Left and Right by Two Functions of $y$</td>
<td>$x$</td>
<td>$y$</td>
<td>you have a region $R$ whose boundary you can parameterize with a single function of $t$, $(x(t), y(t))$ for $t_{\text{low}} \leq t \leq t_{\text{high}}$.</td>
</tr>
<tr>
<td>$\int_{y_{\text{bottom}}}^{y_{\text{top}}} \int_{x_{\text{low}}(y)}^{x_{\text{high}}(y)} f(x, y) , dx , dy$</td>
<td>$\int_{x_{\text{left}}}^{x_{\text{right}}} \int_{y_{\text{low}}(x)}^{y_{\text{high}}(x)} f(x, y) , dy , dx$</td>
<td>$\int_{t_{\text{low}}}^{t_{\text{high}}} n(x(t), y(t)) y'(t) , dt$</td>
<td></td>
</tr>
</tbody>
</table>

Integrate with respect to $x$ first then $y$ when you have a region $R$ that is bounded on the left and right by two functions of $y$:

Integrate with respect to $y$ first then $x$ when you have a region $R$ that is bounded on the top and bottom by two functions of $x$:

Use Gauss-Green when you have a region $R$ whose boundary you can parameterize with a single function of $t$, $(x(t), y(t))$ for $t_{\text{low}} \leq t \leq t_{\text{high}}$. 

![Region R bounded by functions](image)

![Region R bounded by functions](image)

![Region R bounded by functions](image)